On Bott-Chern cohomology of compact complex surfaces

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ON BOTT-CHERN COHOMOLOGY OF COMPACT COMPLEX SURFACES

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Abstract. We study Bott-Chern cohomology on compact complex non-Kähler surfaces. In particular, we compute such a cohomology for compact complex surfaces in class VII and for compact complex surfaces diffeomorphic to solvmanifolds.

Introduction

For a given complex manifold X, many cohomological invariants can be defined, and many are known for compact complex surfaces.

Among these, one can consider *Bott-Chern and Aeppli cohomologies*. They are defined as follows:

$$H^{\bullet,\bullet}_{BC}(X) \;:=\; \frac{\ker\partial\cap\ker\overline{\partial}}{\operatorname{im}\partial\overline{\partial}} \qquad \text{and} \qquad H^{\bullet,\bullet}_{A}(X) \;:=\; \frac{\ker\partial\overline{\partial}}{\operatorname{im}\partial+\operatorname{im}\overline{\partial}} \;.$$

Note that the identity induces natural maps

$$\begin{tikzcd} & H^{\bullet,\bullet}_{BC}(X) \arrow[dl] \arrow[d] \arrow[dr] & \\ H^{\bullet,\bullet}_{\overline{\partial}}(X) \arrow[dr] & H^{\bullet}_{dR}(X;\mathbb{C}) \arrow[d] & H^{\bullet,\bullet}_{\partial}(X) \arrow[dl] \\ & H^{\bullet,\bullet}_{A}(X) & \end{tikzcd}$$

where $H^{\bullet,\bullet}_{\overline{\partial}}(X)$ denotes the Dolbeault cohomology and $H^{\bullet,\bullet}_{\partial}(X)$ its conjugate, and the maps are morphisms of (graded or bi-graded) vector spaces. For compact Kähler manifolds, the natural map $\bigoplus_{p+q=\bullet} H^{p,q}_{BC}(X) \to H^{\bullet}_{dR}(X;\mathbb{C})$ is an isomorphism.

Assume that X is compact. The Bott-Chern and Aeppli cohomologies are isomorphic to the kernel of suitable 4th-order differential elliptic operators, see [19, §2.b, §2.c]. In particular, they are finite-dimensional vector spaces. In fact, fixed a Hermitian metric g, its associated $\mathbb{C}$-linear Hodge-$*$-operator induces the isomorphism

$$H^{p,q}_{BC}(X) \stackrel{\simeq}{\to} H^{n-q,n-p}_{A}(X) \;,$$

for any $p,q\in\{0,\ldots,n\}$, where n denotes the complex dimension of X. In particular, for any $p,q\in\{0,\ldots,n\}$, one has

$$\dim_{\mathbb{C}} H^{p,q}_{BC}(X) \;=\; \dim_{\mathbb{C}} H^{q,p}_{BC}(X) \;=\; \dim_{\mathbb{C}} H^{n-p,n-q}_{A}(X) \;=\; \dim_{\mathbb{C}} H^{n-q,n-p}_{A}(X) \;.$$

For the Dolbeault cohomology, the Frölicher inequality relates the Hodge numbers and the Betti numbers: for any $k\in\{0,\ldots,2n\}$,

$$\sum_{p+q=k} \dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}(X) \;\geq\; \dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;.$$

Similarly, for Bott-Chern cohomology, the following inequality *à la* Frölicher has been proven in [3, Theorem A]: for any $k\in\{0,\ldots,n\}$,

$$\sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) \;\geq\; 2\,\dim_{\mathbb{C}} H^{k}_{dR}(X;\mathbb{C}) \;.$$

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The equality in the Frölicher inequality characterizes the degeneration of the Frölicher spectral sequence at the first level. This always happens for compact complex surfaces. On the other side, in [3, Theorem B], it is proven that the equality in the inequality *à la* Frölicher for the Bott-Chern cohomology characterizes the validity of the $\partial\overline{\partial}$*-Lemma*, namely, the property that every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact too, [8]. The validity of the $\partial\overline{\partial}$-Lemma implies that the first Betti number is even, which is equivalent to Kählerness for compact complex surfaces. Therefore the positive integer numbers

$$\Delta^k \;:=\; \sum_{p+q=k} \left(\dim_{\mathbb{C}} H^{p,q}_{BC}(X) + \dim_{\mathbb{C}} H^{p,q}_{A}(X)\right) - 2\, b_k \;\in\; \mathbb{N}\;,$$

varying $k \in \{1,2\}$, measure the non-Kählerness of compact complex surfaces X.

Compact complex surfaces are divided in seven classes, according to the Kodaira and Enriques classification, see, e.g., [4]. In this note, we compute Bott-Chern cohomology for some classes of compact complex (non-Kähler) surfaces. In particular, we are interested in studying the relations between Bott-Chern cohomology and de Rham cohomology, looking at the injectivity of the natural map $H^{2,1}_{BC}(X) \to H^3_{dR}(X;\mathbb{C})$. This can be intended as a weak version of the $\partial\overline{\partial}$-Lemma, compare also [10].

More precisely, we start by proving that the non-Kählerness for compact complex surfaces is encoded only in Δ^2, namely, Δ^1 is always zero. This gives a partial answer to a question by T. C. Dinh to the third author.

Theorem 1.1. *Let X be a compact complex surface. Then:*

(i) the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\partial}}(X)$ induced by the identity is injective;

(ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

For compact complex surfaces in class VII, we show the following result, where we denote $h^{p,q}_{BC} := \dim_{\mathbb{C}} H^{p,q}_{BC}(X)$ for $p,q \in \{0,1,2\}$.

Theorem 2.2. *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccccc} & & & & h^{0,0}_{BC}=1 & & & & \\ & & h^{1,0}_{BC}=0 & & & & h^{0,1}_{BC}=0 & & \\ h^{2,0}_{BC}=0 & & & & h^{1,1}_{BC}=b_2+1 & & & & h^{0,2}_{BC}=0 \\ & & h^{2,1}_{BC}=1 & & & & h^{1,2}_{BC}=1 & & \\ & & & & h^{2,2}_{BC}=1\;. & & & & \end{array}$$

According to Theorem 1.1, the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\partial}}(X)$ is injective for any compact complex surface. One is then interested in studying the injectivity of the natural map $H^{2,1}_{BC}(X) \to H^3_{dR}(X;\mathbb{C})$ induced by the identity, at least for compact complex surfaces diffeomorphic to solvmanifolds. In fact, by definition, the property of satisfying the $\partial\overline{\partial}$-Lemma, [8], is equivalent to the natural map $\bigoplus_{p+q=\bullet} H^{p,q}_{BC}(X) \to H^\bullet_{dR}(X;\mathbb{C})$ being injective. Note that, for a compact complex manifold of complex dimension n, the injectivity of the map $H^{n,n-1}_{BC}(X) \to H^{2n-1}_{dR}(X;\mathbb{C})$ implies the $(n-1,n)$-th weak $\partial\overline{\partial}$-Lemma in the sense of J. Fu and S.-T. Yau, [10, Definition 5].

We then compute the Bott-Chern cohomology for compact complex surfaces diffeomorphic to solvmanifolds, according to the list given by K. Hasegawa in [11], see Theorem 4.1. More precisely, we prove that the cohomologies can be computed by using just left-invariant forms. Furthermore, for such complex surfaces, we note that the natural map $H^{2,1}_{BC}(X) \to H^3_{dR}(X;\mathbb{C})$ is injective, see Theorem 4.2.

We note that the above classes do not exhaust the set of compact complex non-Kähler surfaces, the cohomologies of elliptic surfaces being still unknown.

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1. Non-Kählerness of compact complex surfaces and Bott-Chern cohomology

We recall that, for a compact complex manifold of complex dimension n, for $k \in \{0,\dots,2n\}$, we define the "non-Kählerness" degrees, [3, Theorem A],

$$\Delta^k \;:=\; \sum_{p+q=k} \left(h^{p,q}_{BC} + h^{n-q,n-p}_{BC}\right) - 2\, b_k \in \mathbb{N}\,,.$$

where we use the duality in [19, §2.c] giving $h^{p,q}_{BC} := \dim_{\mathbb{C}} H^{p,q}_{BC}(X) = \dim_{\mathbb{C}} H^{n-q,n-p}_{A}(X)$. According to [3, Theorem B], $\Delta^k = 0$ for any $k \in \{0, \ldots, 2n\}$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma, namely, every ∂-closed $\overline{\partial}$-closed d-exact form is $\partial\overline{\partial}$-exact too. In particular, for a compact complex surface X, the condition $\Delta^1 = \Delta^2 = 0$ is equivalent to X being Kähler, the first Betti number being even, [14, 17, 20], see also [15, Corollaire 5.7], and [5, Theorem 11].

We prove that Δ^1 is always zero for any compact complex surface. In particular, a sufficient and necessary condition for compact complex surfaces to be Kähler is $\Delta^2 = 0$.

Theorem 1.1. *Let X be a compact complex surface. Then:*

(i) the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\partial}}(X)$ induced by the identity is injective;

(ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

Proof. (i) Let $\alpha \in \wedge^{2,1}X$ be such that $[\alpha] = 0 \in H^{2,1}_{\overline{\partial}}(X)$. Let $\beta \in \wedge^{2,0}X$ be such that $\alpha = \overline{\partial}\beta$. Fix a Hermitian metric g on X, and consider the Hodge decomposition of β with respect to the Dolbeault Laplacian $\overline{\square}$: let $\beta = \beta_h + \overline{\partial}^* \lambda$ where $\beta_h \in \wedge^{2,0}X \cap \ker \overline{\square}$, and $\lambda \in \wedge^{2,1}X$. Therefore we have

$$\alpha \;=\; \overline{\partial}\beta \;=\; \overline{\partial}\overline{\partial}^* \lambda \;=\; -\overline{\partial} * \underbrace{(\partial * \lambda)}_{\in \wedge^{2,0}X} \;=\; -\overline{\partial}\,(\partial * \lambda) \;=\; \partial\overline{\partial}\,(*\lambda) \;,$$

where we have used that any $(2,0)$-form is primitive and hence, by the Weil identity, is self-dual. In particular, α is $\partial\overline{\partial}$-exact, so it induces a zero class in $H^{2,1}_{BC}(X)$.

(ii) On the one hand, note that

$$\begin{aligned} H^{1,0}_{BC}(X) \;&=\; \frac{\ker \partial \cap \ker \overline{\partial} \cap \wedge^{1,0}X}{\operatorname{im} \partial\overline{\partial}} \;=\; \ker \partial \cap \ker \overline{\partial} \cap \wedge^{1,0}X \\ &\subseteq\; \ker \overline{\partial} \cap \wedge^{1,0}X \;=\; \frac{\ker \overline{\partial} \cap \wedge^{1,0}X}{\operatorname{im} \overline{\partial}} \;=\; H^{1,0}_{\overline{\partial}}(X)\;. \end{aligned}$$

It follows that

$$\dim_{\mathbb{C}} H^{0,1}_{BC}(X) \;=\; \dim_{\mathbb{C}} H^{1,0}_{BC}(X) \;\leq\; \dim_{\mathbb{C}} H^{1,0}_{\overline{\partial}}(X) \;=\; b_1 - \dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X)\;,$$

where we use that the Frölicher spectral sequence degenerates, hence in particular $b_1 = \dim_{\mathbb{C}} H^{1,0}_{\overline{\partial}}(X) + \dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X)$.

On the other hand, by part (i), we have

$$\dim_{\mathbb{C}} H^{1,2}_{BC}(X) \;=\; \dim_{\mathbb{C}} H^{2,1}_{BC}(X) \;\leq\; \dim_{\mathbb{C}} H^{2,1}_{\overline{\partial}}(X) \;=\; \dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X)\;,$$

where we use the Kodaira and Serre duality $H^{2,1}_{\overline{\partial}}(X) \simeq H^1(X;\Omega^2_X) \simeq H^1(X;\mathcal{O}_X) \simeq H^{0,1}_{\overline{\partial}}(X)$.

By summing up, we get

$$\begin{aligned} \Delta^1 \;&=\; \dim_{\mathbb{C}} H^{0,1}_{BC}(X) + \dim_{\mathbb{C}} H^{1,0}_{BC}(X) + \dim_{\mathbb{C}} H^{1,2}_{BC}(X) + \dim_{\mathbb{C}} H^{2,1}_{BC}(X) - 2\,b_1 \\ &\leq\; 2\,\left(b_1 - \dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X) + \dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X) - b_1\right) \;=\; 0\;, \end{aligned}$$

concluding the proof. □

2. Class VII surfaces

In this section, we compute Bott-Chern cohomology for compact complex surfaces in class VII.

Let X be a compact complex surface. By Theorem 1.1, the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\partial}}(X)$ is always injective. Consider now the case when X is in class VII. If X is minimal, we prove that the same holds for cohomology with values in a line bundle. We will also prove that the natural map $H^{1,2}_{BC}(X) \to H^{1,2}_{\overline{\partial}}(X)$ is not injective.

Proposition 2.1. *Let X be a compact complex surface in class VII$_0$. Let $L \in H^1(X;\mathbb{C}^*) = \mathrm{Pic}^0(X)$. The natural map $H^{2,1}_{BC}(X;L) \to H^{2,1}_{\overline{\partial}}(X;L)$ induced by the identity is injective.*

Proof. Let $\alpha \in \wedge^{2,1}X \otimes L$ be a $\overline{\partial}_L$-exact $(2,1)$-form. We need to prove that α is $\partial_L\overline{\partial}_L$-exact too. Consider $\alpha = \overline{\partial}_L\vartheta$, where $\vartheta \in \wedge^{2,0}X \otimes L$. In particular, $\partial_L\vartheta = 0$, hence $\bar{\vartheta}$ defines a class in $H^{0,2}_{\overline{\partial}}(X;L)$. Note that $H^{0,2}_{\overline{\partial}}(X;L) \simeq H^2(X;\mathcal{O}_X(L)) \simeq H^0(X;K_X \otimes L^{-1}) = \{0\}$ for surfaces of class VII$_0$, [9, Remark 2.21]. It follows that $\bar{\vartheta} = -\overline{\partial}_L\bar{\eta}$ for some $\eta \in \wedge^{1,0}X \otimes L$. Hence $\alpha = \partial_L\overline{\partial}_L\eta$, that is, α is $\partial_L\overline{\partial}_L$-exact. □

We now compute the Bott-Chern cohomology of class VII surfaces.

Theorem 2.2. *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccc} & & & h^{0,0}_{BC}=1 & & & \\ & & h^{1,0}_{BC}=0 & & h^{0,1}_{BC}=0 & & \\ & h^{2,0}_{BC}=0 & & h^{1,1}_{BC}=b_2+1 & & h^{0,2}_{BC}=0 & \\ & & h^{2,1}_{BC}=1 & & h^{1,2}_{BC}=1 & & \\ & & & h^{2,2}_{BC}=1\,. & & & \end{array}$$

Proof. It holds $H^{1,0}_{BC}(X) = \frac{\ker\partial\cap\ker\overline{\partial}\cap\wedge^{1,0}X}{\mathrm{im}\,\partial\overline{\partial}} = \ker\partial\cap\ker\overline{\partial}\cap\wedge^{1,0}X \subseteq \ker\overline{\partial}\cap\wedge^{1,0}X = \frac{\ker\overline{\partial}\cap\wedge^{1,0}X}{\mathrm{im}\,\overline{\partial}} = H^{1,0}_{\overline{\partial}}(X) = \{0\}$ hence $h^{1,0}_{BC} = h^{0,1}_{BC} = 0$.

On the other side, by Theorem 1.1, $0 = \Delta^1 = 2\left(h^{1,0}_{BC} + h^{2,1}_{BC} - b_1\right) = 2\left(h^{2,1}_{BC} - 1\right)$ hence $h^{2,1}_{BC} = h^{1,2}_{BC} = 1$.

Similarly, it holds $H^{2,0}_{BC}(X) = \frac{\ker\partial\cap\ker\overline{\partial}\cap\wedge^{2,0}X}{\mathrm{im}\,\partial\overline{\partial}} = \ker\partial\cap\ker\overline{\partial}\cap\wedge^{2,0}X \subseteq \ker\overline{\partial}\cap\wedge^{2,0}X = \frac{\ker\overline{\partial}\cap\wedge^{2,0}X}{\mathrm{im}\,\overline{\partial}} = H^{2,0}_{\overline{\partial}}(X) = \{0\}$ hence $h^{2,0}_{BC} = h^{0,2}_{BC} = 0$.

Note that, from [3, Theorem A], we have $0 \le \Delta^2 = 2\left(h^{2,0}_{BC} + h^{1,1}_{BC} + h^{0,2}_{BC} - b_2\right) = 2\left(h^{1,1}_{BC} - b_2\right)$ hence $h^{1,1}_{BC} \ge b_2$. More precisely, from [3, Theorem B] and Theorem 1.1, we have that $h^{1,1}_{BC} = b_2$ if and only if $\Delta^2 = 0$ if and only if X satisfies the $\partial\overline{\partial}$-Lemma, in fact X is Kähler, which is not the case.

Finally, we prove that $h^{1,1}_{BC} = b_2 + 1$. Consider the following exact sequences from [21, Lemma 2.3]. More precisely, the sequence

$$0 \to \frac{\mathrm{im}\,\mathrm{d}\cap\wedge^{1,1}X}{\mathrm{im}\,\partial\overline{\partial}} \to H^{1,1}_{BC}(X) \to \mathrm{im}\left(H^{1,1}_{BC}(X) \to H^2_{dR}(X;\mathbb{C})\right) \to 0$$

is clearly exact. Furthermore, fix a Gauduchon metric g. Denote by $\omega := g(J\cdot,\cdot\cdot)$ the $(1,1)$-form associated to g, where J denotes the integrable almost-complex structure. By definition of g being Gauduchon, we have $\partial\overline{\partial}\omega = 0$. The sequence

$$0 \to \frac{\mathrm{im}\,\mathrm{d}\cap\wedge^{1,1}X}{\mathrm{im}\,\partial\overline{\partial}} \stackrel{\langle\cdot|\omega\rangle}{\to} \mathbb{C}$$

is exact. Indeed, firstly note that for $\eta = \partial\overline{\partial}f \in \mathrm{im}\,\partial\overline{\partial}\cap\wedge^{1,1}X$, we have

$$\langle\eta|\omega\rangle = \int_X \partial\overline{\partial}f\wedge\overline{*\omega} = \int_X \partial\overline{\partial}f\wedge\omega = \int_X f\,\partial\overline{\partial}\omega = 0$$

by applying twice the Stokes theorem. Then, we recall the argument in [21, Lemma 2.3(ii)] for proving that the map

$$\langle\cdot|\omega\rangle : \frac{\mathrm{im}\,\mathrm{d}\cap\wedge^{1,1}X}{\mathrm{im}\,\partial\overline{\partial}} \to \mathbb{C}$$

is injective. Take $\alpha = \mathrm{d}\,\beta \in \mathrm{im}\,\mathrm{d}\cap\wedge^{1,1}X\cap\ker\langle\cdot|\omega\rangle$. Then

$$\langle\Lambda\alpha|1\rangle = \langle\alpha|\omega\rangle = 0\,,$$

where Λ is the adjoint operator of $\omega\wedge\cdot$ with respect to $\langle\cdot|\cdot\cdot\rangle$. Then $\Lambda\alpha \in \ker\langle\cdot|1\rangle = \mathrm{im}\,\Lambda\partial\overline{\partial}$, by extending [16, Corollary 7.2.9] by $\mathbb{C}$-linearity. Take $u \in \mathcal{C}^\infty(X;\mathbb{C})$ such that $\Lambda\alpha = \Lambda\partial\overline{\partial}u$. Then, by defining $\alpha' := \alpha - \partial\overline{\partial}u$, we have $[\alpha'] = [\alpha] \in \frac{\mathrm{im}\,\mathrm{d}\cap\wedge^{1,1}X}{\mathrm{im}\,\partial\overline{\partial}}$, and $\Lambda\alpha' = 0$, and $\alpha' = \mathrm{d}\,\beta'$ where $\beta' := \beta - \overline{\partial}u$. In particular, α' is primitive. Since α' is primitive and of type $(1,1)$, then it is anti-self-dual by the Weil identity. Then

$$\|\alpha'\|^2 = \langle\alpha'|\alpha'\rangle = \int_X \alpha'\wedge\overline{*\alpha'} = -\int_X \alpha'\wedge\overline{\alpha'} = -\int_X \mathrm{d}\,\beta'\wedge\mathrm{d}\,\overline{\beta'} = -\int_X \mathrm{d}\left(\beta'\wedge\mathrm{d}\,\overline{\beta'}\right) = 0$$

and hence $\alpha' = 0$, and therefore $[\alpha] = 0$.

Since the space $\frac{\mathrm{im}\,\mathrm{d}\cap\wedge^{1,1}X}{\mathrm{im}\,\partial\overline{\partial}}$ is finite-dimensional, being a sub-space of $H^{1,1}_{BC}(X)$, and since the space $\mathrm{im}\left(H^{1,1}_{BC}(X)\to H^2_{dR}(X;\mathbb{C})\right)$ is finite-dimensional, being a sub-space of $H^2_{dR}(X;\mathbb{C})$, we get that

$$\dim_{\mathbb{C}}\frac{\mathrm{im}\,\mathrm{d}\cap\wedge^{1,1}X}{\mathrm{im}\,\partial\overline{\partial}} \le \dim_{\mathbb{C}}\mathbb{C} = 1\,,$$

and hence

$$b_2 \;<\; \dim_{\mathbb{C}} H^{1,1}_{BC}(X) \;=\; \dim_{\mathbb{C}} \operatorname{im}\left(H^{1,1}_{BC}(X) \to H^2_{dR}(X;\mathbb{C})\right) + \dim_{\mathbb{C}} \frac{\operatorname{im} \mathrm{d} \cap \wedge^{1,1} X}{\operatorname{im} \partial\bar\partial} \;\leq\; b_2+1 \;.$$

We get that $\dim_{\mathbb{C}} H^{1,1}_{BC}(X) = b_2+1$. $\square$

Finally, we prove that the natural map $H^{1,2}_{BC}(X) \to H^{1,2}_{\bar\partial}(X)$ is not injective.

Proposition 2.3. *Let X be a compact complex surface in class VII. Then the natural map $H^{1,2}_{BC}(X) \to H^{1,2}_{\bar\partial}(X)$ induced by the identity is the zero map and not an isomorphism.*

Proof. Note that, for class VII surfaces, the pluri-genera are zero. In particular, $H^{1,2}_{\bar\partial}(X) \simeq H^{1,0}_{\bar\partial}(X) = \{0\}$, by Kodaira and Serre duality. By Theorem 2.2, one has $H^{1,2}_{BC}(X) \neq \{0\}$. $\square$

2.1. **Cohomologies of Calabi-Eckmann surface.** In this section, as an explicit example, we list the representatives of the cohomologies of a compact complex surface in class VII: namely, we consider the Calabi-Eckmann structure on the differentiable manifolds underlying the Hopf surfaces.

Consider the differentiable manifold $X := \mathbb{S}^1 \times \mathbb{S}^3$. As a Lie group, $\mathbb{S}^3 = SU(2)$ has a global left-invariant co-frame $\left\{e^1, e^2, e^3\right\}$ such that $\mathrm{d}\, e^1 = -2e^2 \wedge e^3$ and $\mathrm{d}\, e^2 = 2e^1 \wedge e^3$ and $\mathrm{d}\, e^3 = -2e^1 \wedge e^2$. Hence, we consider a global left-invariant co-frame $\left\{f, e^1, e^2, e^3\right\}$ on X with structure equations

$$\begin{cases} \mathrm{d}\, f &=& 0 \\ \mathrm{d}\, e^1 &=& -2\, e^2 \wedge e^3 \\ \mathrm{d}\, e^2 &=& 2\, e^1 \wedge e^3 \\ \mathrm{d}\, e^3 &=& -2\, e^1 \wedge e^2 \end{cases} .$$

Consider the left-invariant almost-complex structure defined by the $(1,0)$-forms

$$\begin{cases} \varphi^1 &:=& e^1 + \mathrm{i}\, e^2 \\ \varphi^2 &:=& e^3 + \mathrm{i}\, f \end{cases} .$$

By computing the complex structure equations, we get

$$\begin{cases} \partial\varphi^1 &=& \mathrm{i}\, \varphi^1 \wedge \varphi^2 \\ \partial\varphi^2 &=& 0 \end{cases} \qquad \text{and} \qquad \begin{cases} \bar\partial\varphi^1 &=& \mathrm{i}\, \varphi^1 \wedge \bar\varphi^2 \\ \bar\partial\varphi^2 &=& -\mathrm{i}\, \varphi^1 \wedge \bar\varphi^1 \end{cases} .$$

We note that the almost-complex structure is in fact integrable.

The manifold X is a compact complex manifold not admitting Kähler metrics. It is bi-holomorphic to the complex manifold $M_{0,1}$ considered by Calabi and Eckmann, [6], see [18, Theorem 4.1].

Consider the Hermitian metric g whose associated $(1,1)$-form is

$$\omega \;:=\; \frac{\mathrm{i}}{2} \sum_{j=1}^{2} \varphi^j \wedge \bar\varphi^j \;.$$

As for the de Rham cohomology, from the Künneth formula we get

$$H^\bullet_{dR}(X;\mathbb{C}) \;=\; \mathbb{C}\left\langle 1 \right\rangle \oplus \mathbb{C}\left\langle \varphi^2 - \bar\varphi^2 \right\rangle \oplus \mathbb{C}\left\langle \varphi^{12\bar1} - \varphi^{1\bar1\bar2} \right\rangle \oplus \mathbb{C}\left\langle \varphi^{12\bar1\bar2} \right\rangle \;,$$

(where, here and hereafter, we shorten, e.g., $\varphi^{12\bar1} := \varphi^1 \wedge \varphi^2 \wedge \bar\varphi^1$).

By [12, Appendix II, Theorem 9.5], one has that a model for the Dolbeault cohomology is given by

$$H^{\bullet,\bullet}_{\bar\partial}(X) \;\simeq\; \bigwedge \left\langle x_{2,1},\, x_{0,1} \right\rangle \;,$$

where $x_{i,j}$ is an element of bi-degree (i,j). In particular, we recover that the Hodge numbers $\left\{h^{p,q}_{\bar\partial} := \dim_{\mathbb{C}} H^{p,q}_{\bar\partial}(X)\right\}_{p,q\in\{0,1,2\}}$ are

$$\begin{array}{ccccc} & & h^{0,0}_{\bar\partial} = 1 & & \\ & h^{1,0}_{\bar\partial} = 0 & & h^{0,1}_{\bar\partial} = 1 & \\ h^{2,0}_{\bar\partial} = 0 & & h^{1,1}_{\bar\partial} = 0 & & h^{0,2}_{\bar\partial} = 0 \\ & h^{2,1}_{\bar\partial} = 1 & & h^{1,2}_{\bar\partial} = 0 & \\ & & h^{2,2}_{\bar\partial} = 1 & & \end{array} .$$

We note that the sub-complex

$$\iota \colon \bigwedge \langle \varphi^1,\, \varphi^2,\, \bar\varphi^1,\, \bar\varphi^2 \rangle \hookrightarrow \wedge^{\bullet,\bullet} X$$

is such that $H_{\overline{\partial}}(\iota)$ is an isomorphism. More precisely, we get

$$H^{\bullet,\bullet}_{\overline{\partial}}(X) \quad = \quad \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\left\langle \left[\varphi^{\bar 2}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{12\bar 1}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{12\bar 1\bar 2}\right] \right\rangle \;,$$

where we have listed the harmonic representatives with respect to the Dolbeault Laplacian of g.

By [2, Theorem 1.3, Proposition 2.2], we have also $H_{BC}(\iota)$ isomorphism. In particular, we get

$$H^{\bullet,\bullet}_{BC}(X) \quad = \quad \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\left\langle \left[\varphi^{1\bar 1}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{12\bar 1}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{1\bar 1\bar 2}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{12\bar 1\bar 2}\right] \right\rangle \;,$$

where we have listed the harmonic representatives with respect to the Bott-Chern Laplacian of g.

By [19, §2.c], we have

$$H^{\bullet,\bullet}_{A}(X) \quad = \quad \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\left\langle \left[\varphi^{2}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{\bar 2}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{2\bar 2}\right] \right\rangle \oplus \mathbb{C}\left\langle \left[\varphi^{12\bar 1\bar 2}\right] \right\rangle \;,$$

where we have listed the harmonic representatives with respect to the Aeppli Laplacian of g.

Note in particular that the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\partial}}(X)$ induced by the identity is an isomorphism, and that the natural map $H^{2,1}_{BC}(X) \to H^{3}_{dR}(X;\mathbb{C})$ induced by the identity is injective.

3. Complex surfaces diffeomorphic to solvmanifolds

Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma\backslash G$. By [11, Theorem 1], X is *(A)* either a complex torus, *(B)* or a hyperelliptic surface, *(C)* or a Inoue surface of type $\mathcal{S}_M$, *(D)* or a primary Kodaira surface, *(E)* or a secondary Kodaira surface, *(F)* or a Inoue surface of type $\mathcal{S}^{\pm}$, and, as such, it is endowed with a left-invariant complex structure.

In each case, we recall the structure equations of the group G, see [11]. More precisely, take a basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebra $\mathfrak{g}$ naturally associated to G. We have the following commutation relations, according to [11]:

(A) differentiable structure underlying a *complex torus*:

$$[e_j, e_k] = 0 \qquad \text{for any } j,k \in \{1,2,3,4\} \;;$$

(hereafter, we write only the non-trivial commutators);

(B) differentiable structure underlying a *hyperelliptic surface*:

$$[e_1, e_4] = e_2 \;, \quad [e_2, e_4] = -e_1 \;;$$

(C) differentiable structure underlying a *Inoue surface of type $\mathcal{S}_M$*:

$$[e_1, e_4] = -\alpha\, e_1 + \beta\, e_2 \;, \quad [e_2, e_4] = -\beta\, e_1 - \alpha\, e_2 \;, \quad [e_3, e_4] = 2\alpha\, e_3 \;,$$

where $\alpha \in \mathbb{R}\setminus\{0\}$ and $\beta \in \mathbb{R}$;

(D) differentiable structure underlying a *primary Kodaira surface*:

$$[e_1, e_2] = -e_3 \;;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$[e_1, e_2] = -e_3 \;, \quad [e_1, e_4] = e_2 \;, \quad [e_2, e_4] = -e_1 \;;$$

(F) differentiable structure underlying a *Inoue surface of type $\mathcal{S}^{\pm}$*:

$$[e_2, e_3] = -e_1 \;, \quad [e_2, e_4] = -e_2 \;, \quad [e_3, e_4] = e_3 \;.$$

Denote by $\left\{e^1, e^2, e^3, e^4\right\}$ the dual basis of $\{e_1, e_2, e_3, e_4\}$. We recall that, for any $\alpha \in \mathfrak{g}^*$, for any $x, y \in \mathfrak{g}$, it holds $\mathrm{d}\,\alpha(x,y) = -\alpha\left([x,y]\right)$. Hence we get the following structure equations:

(A) differentiable structure underlying a *complex torus*:

$$\begin{cases} \mathrm{d}\, e^1 &=& 0 \\ \mathrm{d}\, e^2 &=& 0 \\ \mathrm{d}\, e^3 &=& 0 \\ \mathrm{d}\, e^4 &=& 0 \end{cases} \;;$$

(B) differentiable structure underlying a *hyperelliptic surface*:

$$\begin{cases} \mathrm{d}\, e^1 &=& e^2 \wedge e^4 \\ \mathrm{d}\, e^2 &=& -e^1 \wedge e^4 \\ \mathrm{d}\, e^3 &=& 0 \\ \mathrm{d}\, e^4 &=& 0 \end{cases} ;$$

(C) differentiable structure underlying a *Inoue surface of type $\mathcal{S}_M$*:

$$\begin{cases} \mathrm{d}\, e^1 &=& \alpha\, e^1 \wedge e^4 + \beta\, e^2 \wedge e^4 \\ \mathrm{d}\, e^2 &=& -\beta\, e^1 \wedge e^4 + \alpha\, e^2 \wedge e^4 \\ \mathrm{d}\, e^3 &=& -2\alpha\, e^3 \wedge e^4 \\ \mathrm{d}\, e^4 &=& 0 \end{cases} ;$$

(D) differentiable structure underlying a *primary Kodaira surface*:

$$\begin{cases} \mathrm{d}\, e^1 &=& 0 \\ \mathrm{d}\, e^2 &=& 0 \\ \mathrm{d}\, e^3 &=& e^1 \wedge e^2 \\ \mathrm{d}\, e^4 &=& 0 \end{cases} ;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$\begin{cases} \mathrm{d}\, e^1 &=& e^2 \wedge e^4 \\ \mathrm{d}\, e^2 &=& -e^1 \wedge e^4 \\ \mathrm{d}\, e^3 &=& e^1 \wedge e^2 \\ \mathrm{d}\, e^4 &=& 0 \end{cases} ;$$

(F) differentiable structure underlying a *Inoue surface of type $\mathcal{S}^{\pm}$*:

$$\begin{cases} \mathrm{d}\, e^1 &=& e^2 \wedge e^3 \\ \mathrm{d}\, e^2 &=& e^2 \wedge e^4 \\ \mathrm{d}\, e^3 &=& -e^3 \wedge e^4 \\ \mathrm{d}\, e^4 &=& 0 \end{cases} .$$

In cases (A), (B), (C), (D), (E), consider the G-left-invariant almost-complex structure J on X defined by

$$Je_1 := e_2 \quad \text{and } Je_2 := -e_1 \quad \text{and} \quad Je_3 := e_4 \quad \text{and } Je_4 := -e_3 .$$

Consider the G-left-invariant $(1,0)$-forms

$$\begin{cases} \varphi^1 &:=& e^1 + \mathrm{i}\, e^2 \\ \varphi^2 &:=& e^3 + \mathrm{i}\, e^4 \end{cases} .$$

In case (F), consider the G-left-invariant almost-complex structure J on X defined by

$$Je_1 := e_2 \quad \text{and } Je_2 := -e_1 \quad \text{and} \quad Je_3 := e_4 - q\, e_2 \quad \text{and } Je_4 := -e_3 - q\, e_1 ,$$

where $q \in \mathbb{R}$. Consider the G-left-invariant $(1,0)$-forms

$$\begin{cases} \varphi^1 &:=& e^1 + \mathrm{i}\, e^2 + \mathrm{i}\; q\, e^4 \\ \varphi^2 &:=& e^3 + \mathrm{i}\, e^4 \end{cases} .$$

With respect to the G-left-invariant coframe $\left\{\varphi^1, \varphi^2\right\}$ for the holomorphic tangent bundle $T^{1,0}\, \Gamma\backslash G$, we have the following structure equations. (As for notation, we shorten, e.g., $\varphi^{1\bar{2}} := \varphi^1 \wedge \bar{\varphi}^2$.)

(A) *torus*:

$$\begin{cases} \mathrm{d}\, \varphi^1 &=& 0 \\ \mathrm{d}\, \varphi^2 &=& 0 \end{cases}$$

(B) *hyperelliptic surface*:

$$\begin{cases} d\,\varphi^1 &= -\frac{1}{2}\,\varphi^{12} + \frac{1}{2}\,\varphi^{1\bar{2}} \\ d\,\varphi^2 &= 0 \end{cases}$$

(C) *Inoue surface $\mathcal{S}_M$*:

$$\begin{cases} d\,\varphi^1 &= \frac{\alpha - \mathrm{i}\,\beta}{2\,\mathrm{i}}\,\varphi^{12} - \frac{\alpha - \mathrm{i}\,\beta}{2\,\mathrm{i}}\,\varphi^{1\bar{2}} \\ d\,\varphi^2 &= -\,\mathrm{i}\,\alpha\,\varphi^{2\bar{2}} \end{cases}$$

(where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$);

(D) *primary Kodaira surface*:

$$\begin{cases} d\,\varphi^1 &= 0 \\ d\,\varphi^2 &= \frac{\mathrm{i}}{2}\,\varphi^{1\bar{1}} \end{cases}$$

(E) *secondary Kodaira surface*:

$$\begin{cases} d\,\varphi^1 &= -\frac{1}{2}\,\varphi^{12} + \frac{1}{2}\,\varphi^{1\bar{2}} \\ d\,\varphi^2 &= \frac{\mathrm{i}}{2}\,\varphi^{1\bar{1}} \end{cases}$$

(F) *Inoue surface $\mathcal{S}^{\pm}$*:

$$\begin{cases} d\,\varphi^1 &= \frac{1}{2\,\mathrm{i}}\,\varphi^{12} + \frac{1}{2\,\mathrm{i}}\,\varphi^{2\bar{1}} + \frac{q\,\mathrm{i}}{2}\,\varphi^{2\bar{2}} \\ d\,\varphi^2 &= \frac{1}{2\,\mathrm{i}}\,\varphi^{2\bar{2}} \end{cases} .$$

4. Cohomologies of complex surfaces diffeomorphic to solvmanifolds

In this section, we compute the Dolbeault and Bott-Chern cohomologies of the compact complex surfaces diffeomorphic to a solvmanifold.

We prove the following theorem.

Theorem 4.1. *Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma\backslash G$; denote the Lie algebra of G by $\mathfrak{g}$. Then the inclusion $\left(\wedge^{\bullet,\bullet}\mathfrak{g}^*,\, \partial,\, \overline{\partial}\right) \hookrightarrow \left(\wedge^{\bullet,\bullet}X,\, \partial,\, \overline{\partial}\right)$ induces an isomorphism both in Dolbeault and in Bott-Chern cohomologies. In particular, the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies and the degrees of non-Kählerness are summarized in Table 5.*

Proof. Firstly, we compute the cohomologies of the sub-complex of G-left-invariant forms. The computations are straightforward from the structure equations.

	(A) torus				(B) hyperelliptic				(C) Inoue $\mathcal{S}_M$			
$(\mathbf{p},\mathbf{q})$	$H^{p,q}_{\overline{\partial}}$	$\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}$	$H^{p,q}_{BC}$	$\dim_{\mathbb{C}} H^{p,q}_{BC}$	$H^{p,q}_{\overline{\partial}}$	$\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}$	$H^{p,q}_{BC}$	$\dim_{\mathbb{C}} H^{p,q}_{BC}$	$H^{p,q}_{\overline{\partial}}$	$\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}$	$H^{p,q}_{BC}$	$\dim_{\mathbb{C}} H^{p,q}_{BC}$
$(\mathbf{0},\mathbf{0})$	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1
$(\mathbf{1},\mathbf{0})$	$\langle\varphi^1,\, \varphi^2\rangle$	2	$\langle\varphi^1,\, \varphi^2\rangle$	2	$\langle\varphi^2\rangle$	1	$\langle\varphi^2\rangle$	1	$\langle 0\rangle$	0	$\langle 0\rangle$	0
$(\mathbf{0},\mathbf{1})$	$\langle\varphi^{\bar{1}},\, \varphi^{\bar{2}}\rangle$	2	$\langle\varphi^{\bar{1}},\, \varphi^{\bar{2}}\rangle$	2	$\langle\varphi^{\bar{2}}\rangle$	1	$\langle\varphi^{\bar{2}}\rangle$	1	$\langle\varphi^{\bar{2}}\rangle$	1	$\langle 0\rangle$	0
$(\mathbf{2},\mathbf{0})$	$\langle\varphi^{12}\rangle$	1	$\langle\varphi^{12}\rangle$	1	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0
$(\mathbf{1},\mathbf{1})$	$\langle\varphi^{1\bar{1}},\, \varphi^{1\bar{2}},\, \varphi^{2\bar{1}},\, \varphi^{2\bar{2}}\rangle$	4	$\langle\varphi^{1\bar{1}},\, \varphi^{1\bar{2}},\, \varphi^{2\bar{1}},\, \varphi^{2\bar{2}}\rangle$	4	$\langle\varphi^{1\bar{1}},\, \varphi^{2\bar{2}}\rangle$	2	$\langle\varphi^{1\bar{1}},\, \varphi^{2\bar{2}}\rangle$	2	$\langle 0\rangle$	0	$\langle\varphi^{2\bar{2}}\rangle$	1
$(\mathbf{0},\mathbf{2})$	$\langle\varphi^{\bar{1}\bar{2}}\rangle$	1	$\langle\varphi^{\bar{1}\bar{2}}\rangle$	1	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0
$(\mathbf{2},\mathbf{1})$	$\langle\varphi^{12\bar{1}},\, \varphi^{12\bar{2}}\rangle$	2	$\langle\varphi^{12\bar{1}},\, \varphi^{12\bar{2}}\rangle$	2	$\langle\varphi^{12\bar{1}}\rangle$	1	$\langle\varphi^{12\bar{1}}\rangle$	1	$\langle\varphi^{12\bar{1}}\rangle$	1	$\langle\varphi^{12\bar{1}}\rangle$	1
$(\mathbf{1},\mathbf{2})$	$\langle\varphi^{1\bar{1}\bar{2}},\, \varphi^{2\bar{1}\bar{2}}\rangle$	2	$\langle\varphi^{1\bar{1}\bar{2}},\, \varphi^{2\bar{1}\bar{2}}\rangle$	2	$\langle\varphi^{1\bar{1}\bar{2}}\rangle$	1	$\langle\varphi^{1\bar{1}\bar{2}}\rangle$	1	$\langle 0\rangle$	0	$\langle\varphi^{1\bar{1}\bar{2}}\rangle$	1
$(\mathbf{2},\mathbf{2})$	$\langle\varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle\varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle\varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle\varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle\varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle\varphi^{12\bar{1}\bar{2}}\rangle$	1

Table 1. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

$(\mathbf{p},\mathbf{q})$	(D) primary Kodaira $H^{p,q}_{\overline{\partial}}$	$\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}$	$H^{p,q}_{BC}$	$\dim_{\mathbb{C}} H^{p,q}_{BC}$	(E) secondary Kodaira $H^{p,q}_{\overline{\partial}}$	$\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}$	$H^{p,q}_{BC}$	$\dim_{\mathbb{C}} H^{p,q}_{BC}$	(F) Inoue $\mathcal{S}_{\pm}$ $H^{p,q}_{\overline{\partial}}$	$\dim_{\mathbb{C}} H^{p,q}_{\overline{\partial}}$	$H^{p,q}_{BC}$	$\dim_{\mathbb{C}} H^{p,q}_{BC}$
$(\mathbf{0},\mathbf{0})$	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1
$(\mathbf{1},\mathbf{0})$	$\langle \varphi^{1}\rangle$	1	$\langle \varphi^{1}\rangle$	1	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0
$(\mathbf{0},\mathbf{1})$	$\langle \varphi^{\bar{1}},\, \varphi^{\bar{2}}\rangle$	2	$\langle \varphi^{\bar{1}}\rangle$	1	$\langle \varphi^{\bar{2}}\rangle$	1	$\langle 0\rangle$	0	$\langle \varphi^{\bar{2}}\rangle$	1	$\langle 0\rangle$	0
$(\mathbf{2},\mathbf{0})$	$\langle \varphi^{12}\rangle$	1	$\langle \varphi^{12}\rangle$	1	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0
$(\mathbf{1},\mathbf{1})$	$\langle \varphi^{1\bar{2}},\, \varphi^{2\bar{1}}\rangle$	2	$\langle \varphi^{1\bar{1}},\, \varphi^{1\bar{2}},\, \varphi^{2\bar{1}}\rangle$	3	$\langle 0\rangle$	0	$\langle \varphi^{1\bar{1}}\rangle$	1	$\langle 0\rangle$	0	$\langle \varphi^{2\bar{2}}\rangle$	1
$(\mathbf{0},\mathbf{2})$	$\langle \varphi^{\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{\bar{1}\bar{2}}\rangle$	1	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0	$\langle 0\rangle$	0
$(\mathbf{2},\mathbf{1})$	$\langle \varphi^{12\bar{1}},\, \varphi^{12\bar{2}}\rangle$	2	$\langle \varphi^{12\bar{1}},\, \varphi^{12\bar{2}}\rangle$	2	$\langle \varphi^{12\bar{1}}\rangle$	1	$\langle \varphi^{12\bar{1}}\rangle$	1	$\langle \varphi^{12\bar{1}}\rangle$	1	$\langle \varphi^{12\bar{1}}\rangle$	1
$(\mathbf{1},\mathbf{2})$	$\langle \varphi^{2\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}},\, \varphi^{2\bar{1}\bar{2}}\rangle$	2	$\langle 0\rangle$	0	$\langle \varphi^{1\bar{1}\bar{2}}\rangle$	1	$\langle 0\rangle$	0	$\langle \varphi^{1\bar{1}\bar{2}}\rangle$	1
$(\mathbf{2},\mathbf{2})$	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1

TABLE 2. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

$\mathbf{k}$	(A) torus H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	(B) hyperelliptic H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	(C) Inoue $\mathcal{S}_M$ H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$
$\mathbf{0}$	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1
$\mathbf{1}$	$\langle \varphi^{1},\, \varphi^{2},\, \varphi^{\bar{1}},\, \varphi^{\bar{2}}\rangle$	4	$\langle \varphi^{2},\, \varphi^{\bar{2}}\rangle$	2	$\langle \varphi^{2} - \varphi^{\bar{2}}\rangle$	1
$\mathbf{2}$	$\langle \varphi^{12},\, \varphi^{1\bar{1}},\, \varphi^{1\bar{2}},\, \varphi^{2\bar{1}},\, \varphi^{2\bar{2}},\, \varphi^{\bar{1}\bar{2}}\rangle$	6	$\langle \varphi^{1\bar{1}},\, \varphi^{2\bar{2}}\rangle$	2	$\langle 0\rangle$	0
$\mathbf{3}$	$\langle \varphi^{12\bar{1}},\, \varphi^{12\bar{2}},\, \varphi^{1\bar{1}\bar{2}},\, \varphi^{2\bar{1}\bar{2}}\rangle$	4	$\langle \varphi^{12\bar{1}},\, \varphi^{1\bar{1}\bar{2}}\rangle$	2	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}}\rangle$	1
$\mathbf{4}$	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1

TABLE 3. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

$\mathbf{k}$	(D) primary Kodaira H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	(E) secondary Kodaira H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$	(F) Inoue $S^{\pm}$ H^k_{dR}	$\dim_{\mathbb{C}} H^k_{dR}$
$\mathbf{0}$	$\langle 1\rangle$	1	$\langle 1\rangle$	1	$\langle 1\rangle$	1
$\mathbf{1}$	$\langle \varphi^{1},\, \varphi^{\bar{1}},\, \varphi^{2} - \varphi^{\bar{2}}\rangle$	3	$\langle \varphi^{2} - \varphi^{\bar{2}}\rangle$	1	$\langle \varphi^{2} - \varphi^{\bar{2}}\rangle$	1
$\mathbf{2}$	$\langle \varphi^{12},\, \varphi^{1\bar{2}},\, \varphi^{2\bar{1}},\, \varphi^{\bar{1}\bar{2}}\rangle$	4	$\langle 0\rangle$	0	$\langle 0\rangle$	0
$\mathbf{3}$	$\langle \varphi^{12\bar{2}},\, \varphi^{2\bar{1}\bar{2}},\, \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}}\rangle$	3	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}} - q\,\varphi^{12\bar{2}} - \varphi^{1\bar{1}\bar{2}} + q\,\varphi^{2\bar{1}\bar{2}}\rangle$	1
$\mathbf{4}$	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}}\rangle$	1

TABLE 4. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

In Tables 1 and 2 and in Tables 3 and 4, we summarize the results of the computations. The subcomplexes of left-invariant forms are depicted in Figure 1 (each dot represents a generator, vertical arrows depict the $\overline{\partial}$-operator, horizontal arrows depict the ∂-operator, and trivial arrows are not shown.) The dimensions are listed in Table 5.

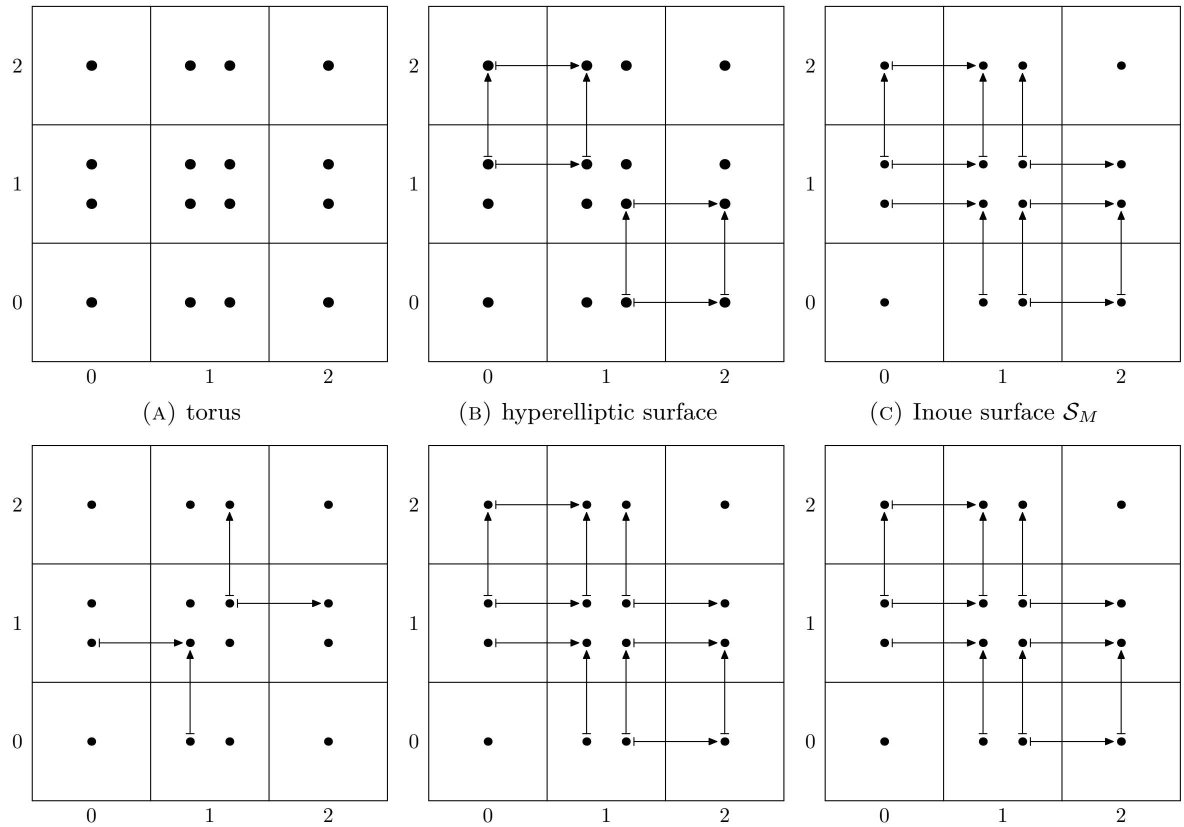
On the one side, recall that the inclusion of left-invariant forms into the space of forms induces an injective map in Dolbeault and Bott-Chern cohomologies, see, e.g., [7, Lemma 9], [1, Lemma 3.6]. On the other side, recall that the Frölicher spectral sequence of a compact complex surface X degenerates at the first level, equivalently, the equalities

$$\dim_{\mathbb{C}} H^{1,0}_{\overline{\partial}}(X) + \dim_{\mathbb{C}} H^{0,1}_{\overline{\partial}}(X) \;=\; \dim_{\mathbb{C}} H^{1}_{dR}(X;\mathbb{C})$$

and

$$\dim_{\mathbb{C}} H^{2,0}_{\overline{\partial}}(X) + \dim_{\mathbb{C}} H^{1,1}_{\overline{\partial}}(X) + \dim_{\mathbb{C}} H^{0,2}_{\overline{\partial}}(X) \;=\; \dim_{\mathbb{C}} H^{2}_{dR}(X;\mathbb{C})$$

hold. By comparing the dimensions in Table 5 with the Betti numbers case by case, we find that the left-invariant forms suffice in computing the Dolbeault cohomology for each case. Then, by [1, Theorem 3.7], see also [2, Theorem 1.3, Theorem 1.6], it follows that also the Bott-Chern cohomology is computed using just left-invariant forms. □



(A) torus (B) hyperelliptic surface (C) Inoue surface $\mathcal{S}_M$

(D) primary Kodaira surface (E) secondary Kodaira surface (F) Inoue surface $\mathcal{S}^{\pm}$

FIGURE 1. The double-complexes of left-invariant forms over 4-dimensional solvmanifolds.

	(A) torus				(B) hyperell				(C) Inoue $\mathcal{S}_M$				(D) prim Kod				(E) sec Kod				(F) Inoue $\mathcal{S}^{\pm}$			
$(\mathbf{p},\mathbf{q})$	$h^{p,q}_{\overline{\partial}}$	$h^{p,q}_{BC}$	b_k	Δ^k	$h^{p,q}_{\overline{\partial}}$	$h^{p,q}_{BC}$	b_k	Δ^k	$h^{p,q}_{\overline{\partial}}$	$h^{p,q}_{BC}$	b_k	Δ^k	$h^{p,q}_{\overline{\partial}}$	$h^{p,q}_{BC}$	b_k	Δ^k	$h^{p,q}_{\overline{\partial}}$	$h^{p,q}_{BC}$	b_k	Δ^k	$h^{p,q}_{\overline{\partial}}$	$h^{p,q}_{BC}$	b_k	Δ^k
$(\mathbf{0},\mathbf{0})$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0
$(\mathbf{1},\mathbf{0})$	2	2	4	0	1	1	2	0	0	0	1	0	1	1	3	0	0	0	1	0	0	0	1	0
$(\mathbf{0},\mathbf{1})$	2	2			1	1			1	0			2	1			1	0			1	0		
$(\mathbf{2},\mathbf{0})$	1	1	6	0	0	0	2	0	0	0	0	2	1	1	4	2	0	0	0	2	0	0	0	2
$(\mathbf{1},\mathbf{1})$	4	4			2	2			0	1			2	3			0	1			0	1		
$(\mathbf{0},\mathbf{2})$	1	1			0	0			0	0			1	1			0	0			0	0		
$(\mathbf{2},\mathbf{1})$	2	2	4	0	1	1	2	0	1	1	1	0	2	2	3	0	1	1	1	0	1	1	1	0
$(\mathbf{1},\mathbf{2})$	2	2			1	1			0	1			1	2			0	1			0	1		
$(\mathbf{2},\mathbf{2})$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0

TABLE 5. Summary of the dimensions of de Rham, Dolbeault, and Bott-Chern cohomologies and of the degree of non-Kählerness for compact complex surfaces diffeomorphic to solvmanifolds.

We prove the following result.

Theorem 4.2. *Let X be a compact complex surface diffeomorphic to a solvmanifold. Then the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\partial}}(X)$ induced by the identity is an isomorphism, and the natural map $H^{2,1}_{BC}(X) \to H^3_{dR}(X;\mathbb{C})$ induced by the identity is injective.*

Proof. By the general result in Theorem 1.1, the natural map $H^{2,1}_{BC}(X) \to H^{2,1}_{\overline{\partial}}(X)$ is injective. In fact, it is an isomorphism as follows from the computations summarized in Tables 1 and 2. As for the injectivity of the natural map $H^{2,1}_{BC}(X) \to H^3_{dR}(X;\mathbb{C})$, it is a straightforward computation from Tables 1 and 2 and Tables 3 and 4.

As an example, we offer an explicit calculation of the injectivity of the map $H^{2,1}_{BC}(X) \to H^3_{dR}(X;\mathbb{C})$ for the Inoue surfaces of type 0, see [13], see also [22]. We will change a little bit the notation. Recall the construction of Inoue surfaces: let $M \in \mathrm{SL}(3;\mathbb{Z})$ be a unimodular matrix having a real eigenvalue $\lambda > 1$ and two complex eigenvalues $\mu \neq \overline{\mu}$. Take a real eigenvector $(\alpha_1, \alpha_2, \alpha_3)$ and an eigenvector $(\beta_1, \beta_2, \beta_3)$ of M. Let $\mathbb{H} = \{z \in \mathbb{C} \mid \mathfrak{Im}\, z > 0\}$; on the product $\mathbb{H} \times \mathbb{C}$ consider the following transformations defined as

$$\begin{aligned} f_0(z,w) &:= (\lambda z,\, \mu w) \\ f_j(z,w) &:= (z+\alpha_j, w+\beta_j) \quad \text{for} \quad j \in \{1,2,3\} . \end{aligned}$$

Denote by Γ_M the group generated by $f_0, \ldots, f_3$; then Γ_M acts in a properly discontinuous way and without fixed points on $\mathbb{H} \times \mathbb{C}$, and $\mathcal{S}_M := \mathbb{H} \times \mathbb{C} / \Gamma_M$ is an Inoue surface of type 0, as in case (C) in [11]. Denoting by $z = x + \mathrm{i}\, y$ and $w = u + \mathrm{i}\, v$, consider the following differential forms on $\mathbb{H} \times \mathbb{C}$:

$$e^1 := \frac{1}{y}\, \mathrm{d}\, x \,, \quad e^2 := \frac{1}{y}\, \mathrm{d}\, y \,, \quad e^3 := \sqrt{y}\, \mathrm{d}\, u \,, \quad e^4 := \sqrt{y}\, \mathrm{d}\, v \,.$$

(Note that e^1 and e^2, and $e^3 \wedge e^4$ are Γ_M-invariant, and consequently they induce global differential forms on $\mathcal{S}_M$.) We obtain

$$\mathrm{d}\, e^1 = e^1 \wedge e^2 \,, \quad \mathrm{d}\, e^2 = 0 \,, \quad \mathrm{d}\, e^3 = \frac{1}{2}\, e^2 \wedge e^3 \,, \quad \mathrm{d}\, e^4 = \frac{1}{2}\, e^2 \wedge e^4 \,.$$

Consider the natural complex structure on $\mathcal{S}_M$ induced by $\mathbb{H} \times \mathbb{C}$. Locally, we have

$$J e^1 = -e^2 \quad \text{and} \quad J e^2 = e^1 \quad \text{and} \quad J e^3 = -e^4 \quad \text{and} \quad J e^4 = e^3 \,.$$

Considering the Γ_M-invariant $(2,1)$-Bott-Chern cohomology of $\mathcal{S}_M$, we obtain that

$$H^{2,1}_{BC}(\mathcal{S}_M) = \mathbb{C} \left\langle \left[e^1 \wedge e^3 \wedge e^4 + \mathrm{i}\, e^2 \wedge e^3 \wedge e^4 \right] \right\rangle \,.$$

Clearly $\overline{\partial} \left(e^1 \wedge e^3 \wedge e^4 + \mathrm{i}\, e^2 \wedge e^3 \wedge e^4 \right) = 0$ and $e^1 \wedge e^3 \wedge e^4 + \mathrm{i}\, e^2 \wedge e^3 \wedge e^4 = e^1 \wedge e^3 \wedge e^4 + \mathrm{i}\, \mathrm{d} \left(e^3 \wedge e^4 \right)$, therefore the de Rham cohomology class $\left[e^1 \wedge e^3 \wedge e^4 + \mathrm{i}\, e^2 \wedge e^3 \wedge e^4 \right] = \left[e^1 \wedge e^3 \wedge e^4 \right] \in H^3_{dR}(\mathcal{S}_M)$ is non-zero. □

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